Relation & function # Theorem:-

$\left[f : X \to Y;\ g : Y \to Z,\ h : Z \to S \right.$

$\left. \Rightarrow (h o g) o f(x) = h o (g o f)(x) \right]$

Ex:- $f : N \to N,\ g : N \to N,\ h : N \to R$

as $f(x) = 2x,\ g(y) = 3y + 4,\ h(z) = \sin z$

Show:- $h o (g o f) = (h o g) o f$

Sol^n:- LHS:- $h o (g o f) \Rightarrow h o (g o f)(x) = h((g o f)(x))$

$\Rightarrow h(g(\underline{f(x)})) \Rightarrow h(g(2x)) = h(3(2x) + 4) = h(6x + 4)$

$\Rightarrow \underline{\sin(6x + 4)} = \underline{LHS}$

RHS:- $(h o g) o f = (h o g)(f(x)) = (h o g)(2x) = h(g(2x)) = h(3(2x + 4))$

$= h(6x + 4) = \sin(6x + 4) = \underline{RHS}$

$\Rightarrow \left[h o (g o f) = (h o g) o f \right]$ H.P.

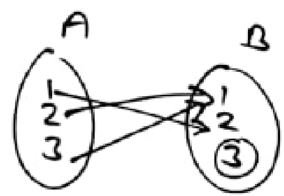
Relation & function

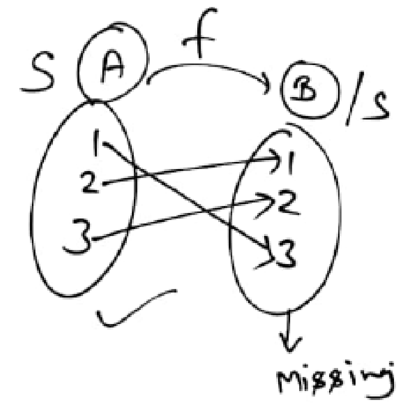
Ex: $S = \{1, 2, 3\}$ $\quad f : S \to S$ $\to$ find f^{-1} exist for which.

a) $f = \{(1,1)(2,2)(3,3)\}$

b) $f = \{(1,2)(2,1)(3,1)\}$

c) $f = \{(1,3)(3,2)(2,1)\}$





Sol^n:- $f^{-1} \to$ one-one | onto

a) $f \to$ one-one — ✓, onto — ✓] f^{-1} exist $\to$ $f^{-1} = \{(1,1)(2,2)(3,3)\}$ Ans

b) $f \to$ one-one — ✗, onto — ✗] f^{-1} doesn't exist

c) $f \to$ one-one $\to$ ✓, onto $\to$ ✓] $f^{-1} \to f^{-1} = \{(3,1)(2,3)(1,2)\}$

Relation & function

$=$ $f, g, h \to$ funⁿ from $R \to R$

show:- i) $(f+g)oh = foh + goh$

solⁿ:- LHS $\to$ $(f+g)\,oh = (f+g)\,oh(x) = [(f+g)](h(x)]$

$= f(h(x)) + g(h(x)) \Rightarrow foh(x) + goh(x) \Rightarrow foh + goh = \text{RHS}$

H.P.

ii) $(f \cdot g)\,oh = (foh)\cdot(goh)$

solⁿ:- LHS:- $(f\cdot g)\,oh = (f\cdot g)\,oh(x) = \{(f\cdot g)\}(h(x))$

$\Rightarrow f(h(x))\cdot g(h(x))$

$= foh(x)\cdot goh(x)$

$\Rightarrow foh \cdot goh = \text{RHS}$

Relation & function

Ex:- $\left[f(x) = \dfrac{4x+3}{6x-4}\right]$, $x \neq \frac{2}{3}$

Show:- $[f \circ f(x) = x]$

find $f^{-1} = ?$

Soln:- $f \circ f(x) = f(f(x))$

$\Rightarrow f\left(\dfrac{4x+3}{6x-4}\right) = \dfrac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$

$\Rightarrow \dfrac{\frac{16x+\cancel{12}+18x-\cancel{12}}{\cancel{6x-4}}}{\frac{\cancel{24x}+18-\cancel{24x}+16}{\cancel{6x-4}}} = \dfrac{34x}{34} = x$

$\Rightarrow$ i.e. $\boxed{f \circ f(x) = x}$

So f^{-1} exist:-

$\left[\& \; f^{-1}(x) = g(y) = \dfrac{4y+3}{6y-4}\right]$ A

$\Rightarrow f^{-1}$ exist:- $[g \circ f(x) = I_x \;\&\; f \circ g(y) = I_y]$

let:- y is an element such that:-

$y = \dfrac{4x+3}{6x-4} \Rightarrow 6xy - 4y = 4x+3$

$\Rightarrow 6xy - 4x = 3 + 4y \Rightarrow x(6y-4) = 3+4y$

$\Rightarrow \left[x = \dfrac{4y+3}{6y-4}\right] \Rightarrow \left[g(y) = \dfrac{4y+3}{6y-4}\right]$

So:- g is the inverse of f if $g \circ f = I_x$ & $f \circ g = I$

$\Rightarrow g \circ f(x) = g[f(x)] = g\left(\dfrac{4x+3}{6x-4}\right) = \dfrac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = x$

$\Rightarrow f \circ g(y) = f(g(y)) \Rightarrow f\left(\dfrac{4y+3}{6y-4}\right)$

$= \dfrac{4\left(\frac{4y+3}{6y-4}\right)+3}{6\left(\frac{4y+3}{6y-4}\right)-4} = y$

$\Rightarrow g \circ f(x) = I_x = x$

$\Rightarrow f \circ g(y) = y = I_y$

Relation & function

Ex:- $f: R \to R$ as $f(x) = 4x+3$

$f^{-1} = ?$

Soln:- f^{-1} exist:-

→ one-one:- $f(x_1) = f(x_2)$

$4x_1 + \cancel{3} = 4x_2 + \cancel{3}$

$\boxed{x_1 = x_2}$

→ onto:- $y = 4x+3 \Rightarrow x = \frac{y-3}{4} \in R$

it is onto.

So:- $f^{-1} \Rightarrow ? \Rightarrow$ let y be an element in R.

$\Rightarrow y = 4x+3 \Rightarrow \left[x = \frac{y-3}{4}\right] \forall x \in R$

$\Rightarrow g(y) = \frac{y-3}{4}$ so:- g can be inverse of f.

Such that $g: R \to R$ as:- $\boxed{g(y) = \frac{y-3}{4}}$

$fog(y) = I_y$ & $gof(x) = I_x$

→ $fog(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = \cancel{4}\left(\frac{y-\cancel{3}}{\cancel{4}}\right) + \cancel{3} = y$

$fog(y) = y = I_y$

→ $gof(x) = g(f(x)) = g(4x+3)$

$= \frac{(\cancel{4}x + \cancel{3}) - \cancel{3}}{\cancel{4}} = x$

→ $\boxed{gof(x) = x = I_x}$

So: inverse of f is $f^{-1} = g(y) = \frac{y-3}{4}$

Relation & function

Ex: $f : R_+ \to [-5, \infty)$

$f(x) = 9x^2 + 6x - 5$

Show that f is invertible

with $f^{-1}(y) = \left(\frac{\sqrt{y+6} - 1}{3}\right)$

Soln:- Let y be an element such that.

$\Rightarrow y = 9x^2 + 6x - 5 = (3x)^2 + 2 \cdot 3x \cdot 1 + (1)^2 - 1 - 5$

$\Rightarrow y = (3x+1)^2 - 6 \Rightarrow y + 6 = (3x+1)^2$

$\Rightarrow 3x + 1 = \sqrt{y+6} \Rightarrow 3x = \sqrt{y+6} - 1$

$\Rightarrow \left[x = \frac{\sqrt{y+6} - 1}{3}\right] \Rightarrow \left\{g(y) = \frac{\sqrt{y+6} - 1}{3}\right\}$

So:- Here g is a fun^n such that $g : [-5, \infty) \to R_+$

as $g(y) = \frac{\sqrt{y+6} - 1}{3}$

$fog(y) = I_y$ & $gof(x) = I_x$

$\to fog(y) = f(g(y)) = f\left[\frac{\sqrt{y+6} - 1}{3}\right] =$

$\left[\Rightarrow 9 \cdot \left(\frac{\sqrt{y+6} - 1}{3}\right)^2 + 6\left[\frac{\sqrt{y+6} - 1}{3}\right] - 5\right] = y$

$fog(x) = x$